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Note

Comment on "On Boundary Conditions for Hyperbolic Difference Schemes"

In a recent paper [1], Gary studies different boundary conditions for the method of lines. In case A the problem

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0 \tag{1}$$

is approximated by

$$\begin{aligned} \frac{du_j}{dt} + \frac{u_{j+1} - u_{j-1}}{2\Delta x} &= 0, \qquad j = 1, 2, \dots, J, \\ \frac{du_0}{dt} + \frac{u_1 - u_0}{\Delta x} &= 0, \\ \frac{du_J}{dt} + \frac{u_J - u_{J-1}}{\Delta x} &= 0, \end{aligned} \tag{2}$$

which can be written in the compact form

$$\frac{dU}{dt} = MU.$$

System (2) is inconsistent in the sense that (1) requires boundary data at $x = 0$, but no such data are given for the semidiscrete scheme. This is an interesting case, because in applications boundary data are sometimes missing, and approximations of type (2) are used.

Gary computes numerically the eigenvalues λ of M and finds that despite the inconsistency, there are no λ-values in the right half plane, but that there appears to be a double or triple eigenvalue at the imaginary axis. In this note we show that this is a natural property.

System (2) is the equivalent of defining a new point at $x = -\Delta x$ and using linear extrapolation for u_{-1},

$$u_{-1} = 2u_0 - u_1.$$

This is an approximation of

$$\frac{\partial^2 u}{\partial x^2} = 0 \qquad \text{at} \quad x = 0, \tag{3}$$

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and apparently problem (1), (3) admits solutions

$$u = \alpha(t - x) + \beta$$

for arbitrary constants α and β. The linear growth is reflected in the double eigenvalue λ mentioned above.

The second equation of (2) is the one causing the instability. Therefore we will investigate the half-plane problem

$$\frac{du_j}{dt} + \frac{u_{j+1} - u_{j+1}}{2\Delta x} = 0, \qquad j = 1, 2, \ldots,$$

$$\frac{du_0}{dt} + \frac{u_1 - u_0}{\Delta x} = 0,$$

$$\sum_{j=0}^{\infty} | u_j |^2 \Delta x < \infty.$$

After a Laplace transformation we obtain

$$\begin{aligned} sv_j + \frac{v_{j+1} - v_{j-1}}{2\Delta x} &= 0, \qquad j = 1, 2, \ldots, \\ sv_0 + \frac{v_1 - v_0}{\Delta x} &= 0 \end{aligned} \tag{4}$$

which has the solution

$$v_j = \sigma(-s\Delta x + ((s\Delta x)^2 + 1)^{1/2})^j, \qquad \operatorname{Re} s > 0,$$

where we get from the boundary condition

$$(((s\Delta x)^2 + 1)^{1/2} - 1)\, \sigma = 0.$$

Therefore we have a nontrivial solution for $(s\Delta x)^2 = 0$, and $s = 0$ is a double eigenvalue to (4). (Actually, $s = 0$ is a generalized eigenvalue, since $v_j \equiv \sigma$.) This is the explanation of the behavior described in [1].

As we noted above, the procedure at the boundary is equivalent to extrapolation. Considering the more general extrapolation formula

$$\delta_+{}^p u_{-1} = 0, \qquad p = 1, 2, \ldots, \qquad \delta_+ u_j = u_{j+1} - u_j\,,$$

we get the condition

$$(-s\Delta x + ((s\Delta x)^2 + 1)^{1/2} - 1)^p = 0$$

or equivalently

$$(s\Delta x)^p = 0.$$

Hence, the multiplicity of the eigenvalue $s = 0$ is p. The conclusion is that the higher the order of extrapolation (i.e., the higher the accuracy), the worse the result, since we get a higher-order polynomial growth with time; see also [2].

The analysis made here is for the quarter space problem $x \geqslant 0$, $t \geqslant 0$. In general, if the boundary $x = 1$ is also taken into consideration, a higher-order growth or an exponential growth may occur. According to the numerical results referred to in [1], a third-order polynomial growth may have been introduced for J odd.

A general theory for the method of lines was recently presented by Strikwerda [3]. According to that theory (and also according to the theory for the fully discretized scheme) condition (5) leads to an instability also for $p = 1$. However, in [2] it is proved for the Lax–Wendroff scheme that for smooth initial data $f(x)$ with $df/dx = 0$ at $x = 0$ we still get a solution which converges when $\Delta x \to 0$ to a limit function satisfying $u(0, t) = f(0)$.

References

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2. B. Gustafsson and H.-O. Kreiss, *J. Computational Physics* **30** (1979), 333–351.
3. J. C. Strikwerda, *J. Computational Physics* **34** (1980), 94–107.

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